A SHORT PROOF OF THE TRIANGLE INEQUALITY FOR THE PRETENTIOUS METRIC

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ABSTRACT. Let $\mathbb{U}(\mathbb{N})$ be the set of all complex-valued multiplicative functions f with $|f(n)| \leq 1$ for all $n \in \mathbb{N}$. The pretentious "metric" on $\mathbb{U}(\mathbb{N})$ over an interval $I \subseteq [1, \infty)$ is defined by

$$\mathbb{D}(f, g; I) := \left(\sum_{p \in I} \frac{1 - \Re(f(p)\overline{g(p)})}{p} \right)^{1/2}.$$

The author gives a short proof of the following triangle inequality:

$$\mathbb{D}(f, g; I) + \mathbb{D}(g, h; I) \geq \mathbb{D}(f, h; I)$$

for any $f, g, h \in \mathbb{U}(\mathbb{N})$.

1. Introduction

Let $\mathbb{U}(\mathbb{N})$ be the set of all complex-valued multiplicative functions f with $|f(n)| \leq 1$ for all $n \in \mathbb{N}$. The pretentious "metric" or "distance" on $\mathbb{U}(\mathbb{N})$ over an interval $I \subseteq [1, \infty)$ is defined by

$$\mathbb{D}(f, g; I) := \left(\sum_{p \in I} \frac{1 - \Re(f(p)\overline{g(p)})}{p} \right)^{1/2}.$$

This quantity satisfies the following triangle inequality [1]:

$$\mathbb{D}(f, g; I) + \mathbb{D}(g, h; I) \geq \mathbb{D}(f, h; I)$$

for any $f, g, h \in \mathbb{U}(\mathbb{N})$. As noted in [1], this follows from the inequality

$$\sqrt{1 - \Re(z)} + \sqrt{1 - \Re(w)} \geq \sqrt{1 - \Re(zw)} \tag{1.1}$$

for $z, w \in \mathbb{C}$ with $|z|, |w| \leq 1$. We shall give a short proof of (1.1).

2. Proof of (1.1)

Squaring both sides of (1.1) we see that it is equivalent to

$$(1 - \Re(z))(1 - \Re(w)) + 2\sqrt{(1 - \Re(z))(1 - \Re(w))} \geq \Im(z)\Im(w),$$

where $z, w \in \mathbb{C}$ with $|z|, |w| \leq 1$. Thus (1.1) follows from

$$(1 - \Re(z))(1 - \Re(w)) + 2\sqrt{(1 - \Re(z))(1 - \Re(w))} \geq |\Im(z)\Im(w)|, \tag{2.1}$$

where $z, w \in \mathbb{C}$ with $|z|, |w| \leq 1$. This inequality is trivial if $z \in \mathbb{R}$ or $w \in \mathbb{R}$. Let $z = Ae^{i\theta}$ and $w = Be^{i\varphi}$, where $A, B \in (0, 1]$ and $\theta, \varphi \in (-\pi, \pi) \setminus \{0\}$. Then we may rewrite (2.1) as

$$(A^{-1} - \cos\theta)(B^{-1} - \cos\varphi) + 2\sqrt{A^{-1}B^{-1}(A^{-1} - \cos\theta)(B^{-1} - \cos\varphi)} \geq |\sin\theta \sin\varphi|. \tag{2.2}$$

Clearly, the left side of (2.2) is a decreasing function of A and B. Hence it is sufficient to prove (2.2) under the additional assumption that $A = B = 1$. In other words, we need only to prove

$$(1-\cos\theta)(1-\cos\varphi)+2\sqrt{(1-\cos\theta)(1-\cos\varphi)} \geq \sqrt{(1-\cos^2\theta)(1-\cos^2\varphi)}$$

whenever $\theta, \varphi \in (-\pi, \pi) \setminus \{0\}$. Canceling out the factor $\sqrt{(1-\cos\theta)(1-\cos\varphi)}$ on both sides we see that the inequality above is equivalent to

$$\sqrt{(1-\cos\theta)(1-\cos\varphi)}+2 \geq \sqrt{(1+\cos\theta)(1+\cos\varphi)}.$$

This inequality is trivial, since the right side never exceeds 2. This completes the proof.

References

[1] A. Granville and K. Soundararajan, *Pretentious multiplicative functions and an inequality for the zeta-function*, Anatomy of Integers, 191197, CRM Proc. Lecture Notes, vol. 46, Amer. Math. Soc., Providence, RI, 2008.

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